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SON CAUCHY AND BOUNDARY VALUE PROBLEMS FOR THE THIRD-ORDER DISCRETE-MULTIPLICATIVE DERIVATIVE EQUATION

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Abstract. In the paper Cauchy and boundary value problem are sonsidered for the third-order discrete-multiplicative derivative differential equation. Using the definition of the discrete multiplicative derivative, the general solution of the considered problem is constructed equation depending on three constants. These constants, which are included in the general solution, are determinated from the given initial or boundary conditions. The analytical formulas are derived for the solutions of the Cauchy and and boundary value problems.

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1 Introduction

The considered here problems have strong applications to the different fields. As example, one can mention the problem of finding the general limit of the numerical series and construction the Fibonacci sequence. For the linear differential equation with a constant coefficients of ordinary discrete additive derivatives (for the difference equations), these problems are referred to Cauchy. The main problem of the discrete analysis is to solve the problems for ordinary or partial derivative equations by replacing the derivatives with differences, at tending the step closer to zero. The question of constructing the general limit of geometric series has lead us to the question of Cauchy for the differential equation with discrete multiplicative derivative. It should be noted that the obtained problem in some cases my be also nonlinear.

Thus, we will mainly deal with the problem for the nonlinear equations. It is possible that even the given conditions are non-linear. In this case, the general solution, which depends on the arbitrary constants (the number of these constants is equal to the order of the equation), is determined, and then the constants entered into the general solution using the initial or Sarah conditions (Aliyev & Fatemi, 2014).

2 Main results

Consider the following equation

$$y_i^{[III]} = f_i, i \geq 0, \tag{1}$$

where f_i, $i \geq 0$ is given real valued series, y_i, $i \geq 0$ is seeking delviered series.

Using the definition of the discrete multiplicative derivative (1) we can write equation (1) in the followin form (Aliyev et al., 2018)

$$y_{i+3} = f_i \frac{y_{i+2}^3}{y_{i+1}^3} y_i, i \geq 0. \tag{2}$$

Let's try to derive an analytical expression for the solution of (2) by giving values to i . For $i = 0$ from (2) we get

$$y_3 = f_0 \frac{y_2^3}{y_1^3} y_0. \tag{3}$$

For $i = 1$ we get

$$y_4 = f_1 \frac{y_3^3}{y_2^3} y_1.$$

Taking into account here (3) one can write

$$y_4 = f_1 \frac{y_3^3}{y_2^3} y_1 = f_1 \cdot \frac{f_0^3 \cdot \frac{y_2^9}{y_1^9} y_0^3}{y_2^3} y_1 = f_1 f_0^3 \frac{y_2^6}{y_1^8} y_0^3. \tag{4}$$

If to set $i = 2$ in (2), then we obtain

$$y_5 = f_2 \frac{y_4^3}{y_3^3} y_2 . y_5 = f_2 \frac{y_4^3}{y_3^3} y_2.$$

Taking into account here (3) and (4) we get

$$y_5 = f_2 \frac{y_4^3}{y_3^3} y_2 = f_2 \cdot \frac{f_1^3 f_0^9 \cdot \frac{y_2^{18}}{y_1^{24}} y_0^9}{f_0^3 \frac{y_2^9}{y_1^9} y_0^3} \cdot y_2 = f_2 f_1^3 f_0^6 \frac{y_2^{10} y_0^6}{y_1^{15}}. \tag{5}$$

Finally, to assure the regularity of the analytical expression of the solution to (2), let's set $i = 3$. Then we get

$$y_6 = f_3 \frac{y_5^3}{y_4^3} y_3.$$

Considering here (3), (4) and (5) we get

$$y_6 = f_3 \frac{y_5^3}{y_4^3} y_3 = f_3 \cdot \frac{f_2^3 f_1^9 f_0^{18} \cdot \frac{y_2^{30} y_0^{18}}{y_1^{45}}}{f_1^3 f_0^9 \frac{y_2^{18} y_0^9}{y_1^{24}}} \cdot$$

$$\cdot f_0 \frac{y_2^3}{y_1^3} y_0 = f_3 f_2^3 f_1^6 f_0^{10} \frac{y_2^{15} y_0^{10}}{y_1^{24}}. \tag{6}$$

Since we see what it is impossible to follow the change in regularity in expressions (3)-(6), we will solve equation (1) in another way.

Let's write equation (1) as follows

$$(y_i^{[II]})^{[1]} = f_i, i \geq 0.$$

Using the definition of the discrete multiplicative derivative we ca rewrite

$$\frac{y_{i+1}^{[II]}}{y_i^{[II]}} = f_i, i \geq 0.$$

Setting values to i we get

$$\frac{y_1^{[II]}}{y_0^{[II]}} = f_0,$$

$$\frac{y_2^{[II]}}{y_1^{[II]}} = f_1,$$

$$\frac{y_3^{[II]}}{y_2^{[II]}} = f_2,$$

$$\ldots$$

$$\frac{y_{i-1}^{[II]}}{y_{i-2}^{[II]}} = f_{i-2},$$

$$\frac{y_i^{[II]}}{y_{i-1}^{[II]}} = f_{i-1}.$$

Multiplying these expressions side-by-side (after excluding similar terms), we get

$$\frac{y_i^{[II]}}{y_0^{[II]}} = \prod_{k=0}^{i-1} f_k,$$

or

$$y_i^{[II]} = y_0^{[II]} \prod_{k=0}^{i-1} f_k, i \geq 1. \tag{7}$$

Introduce the denotations below

$$g_i = g_i\left(y_0^{[II]}, f_k\right) = y_0^{[II]} \prod_{k=0}^{i-1} f_k, i \geq 1. \tag{8}$$

Then equation (7) turns to

$$y_i^{[II]} = g_i, i \geq 1. \tag{9}$$

Thus, we have reduced the third order discrete derivative equation (1) to the second order discrete derivative equation (9) (Bashirov, 2013).

Now by applying the definition of the second-order discrete multiplicative derivative (1) in the second-order discrete multiplicative derivative (9) we get (Bashirov & Riza, 2011)

$$\frac{y_{i+1}^{[I]}}{y_i^{[I]}} = g_i, i \geq 1.$$

Similarly, giving values to i we can write

$$\frac{y_2^{[I]}}{y_1^{[I]}} = g_1,$$

$$\frac{y_3^{[I]}}{y_2^{[I]}} = g_2,$$

$$\frac{y_4^{[I]}}{y_3^{[I]}} = g_3,$$

$$\cdots$$
$$\frac{y_{i-1}^{[I]}}{y_{i-2}^{[I]}} = g_{i-2},$$
$$\frac{y_{i}^{[I]}}{y_{i-1}^{[I]}} = g_{i-1}.$$

Multiplying these expressions side-by-side (after excluding some terms) we get the following equation

$$\frac{y_i^{[I]}}{y_1^{[I]}} = \prod_{s=1}^{i-1} g_s,$$

or

$$y_i^{[I]} = y_1^{[I]} \prod_{s=1}^{i-1} g_s, i \geq 2. \tag{10}$$

Thus, we reduced the third order discrete multiplicative derivative equation (1) to the first order discrete multiplicative derivative equation (10) (Jahanshahi et al., 2011). If to denote here

$$h_i = h_i\left(y_i^{[I]} g_s\right) = y_1^{[I]} \prod_{s=1}^{i-1} g_s, i \geq 2 \tag{11}$$

then the first-order discrete multiplicative derivative equation (10) will take the form

$$y_i^{[I]} = h_i, i \geq 2. \tag{12}$$

Finally, if we apply definition discrete multiplicative derivative to thefirst-order discrete multiplicative derivative equation (12) we get

$$\frac{y_{i+1}}{y_i} = h_i, i \geq 2.$$

Setting values to i we get

$$\frac{y_3}{y_2} = h_2,$$
$$\frac{y_4}{y_3} = h_3,$$
$$\cdots$$
$$\frac{y_{i-1}}{y_{i-2}} = h_{i-2},$$
$$\frac{y_i}{y_{i-1}} = h_{i-1}.$$

By multiplying these expressions side-by-side we obtain

$$\frac{y_i}{y_2} = \prod_{m=2}^{i-1} h_m,$$

or

$$y_i = y_2 \prod_{m=2}^{i-1} h_m, i \geq 3. \tag{13}$$

Thus, for the general solution of the third order of discrete multiplicative derivative equation (1) we get expression (13). Here are defined by (11), are defined by via (7) and are arbitrary constants (Dezin, 1980).

Thus, we proved the following

Theorem 1. *If is a given real valued series, then the general solution to equation* (1) *is of the form* (13). *Here h_i are defined by* (11), *g_i are defined by* (8) *and $y_0^{[II]}, y_1^{[1]}$ and y_2 are arbitrary constants (Hosseini et al., 2017)*

Cauchy problem. Now we set the following boundary conditions

$$y_k = \alpha_k, k = \overline{0,2} \tag{14}$$

to equation (1). Here $\alpha_k, k = \overline{0,2}$ are given real numbers (Jahanshahi et al., 2011) Then the solution to Cauchy problem (1), (14) will be obtained from (13) that is the general solution to (1) (Oskouei & Aliev, 2008). First of all if to consider in (8) that

$$y_0^{[II]} = \frac{y_0 y_2}{y_1^2} = \frac{\alpha_0 \alpha_2}{\alpha_1^2}.$$

The for g_i we get

$$g_i = \frac{\alpha_0 \alpha_2}{\alpha_1^2} \prod_{k=0}^{i-1} f_k, i \geq 1, \tag{15}$$

which does not contain any discretion. Then in (11) considering

$$y_1^{[I]} = \frac{y_2}{y_1} = \frac{\alpha_2}{\alpha_1},$$

we get

$$h_i = \frac{\alpha_2}{\alpha_1} \prod_{s=1}^{i-1} g_s, i \geq 2. \tag{16}$$

Finally, putting the expression (14) for y^2 into (13) for the solution of the Cauchy problem we get

$$y_i = \alpha_2 \prod_{m=2}^{i-1} h_m, i \geq 3. \tag{17}$$

Thus, the followin theorwm is proved.

Theorem 2. *Within the terms of Theorem 1. if α_0, $\alpha_1 \neq 0$ and α_2 are the given real constants, then the solution to the Cauchy problem is of the form* (1), (14), (17). *Here h_i are defined by* (16) *and g_i are defined by* (15).

Boundary value problem. Now we consider the following boundary value problem for the equation (1) at the values $\overline{0, N-3}$ of i

$$y_0^{[II]} = \alpha, y_1^{[I]} = \beta, y_N = \gamma. \tag{18}$$

Then considering the first two conditions from (8) and (11) we get

$$g_i = \alpha \prod_{k=0}^{i-1} f_k, i \geq 1. \tag{19}$$

Here g_i, y_N and $h_i,\ i \geq 2$ are defined unequivocally. Then, taking into account the third condition from (18) in the general solution (13) for y_2 we get linear algebraic equation

$$\gamma = y_N = y_2 \prod_{m=2}^{N-1} h_m. \tag{20}$$

Let's define y_2 from (20)

$$y_2 = \frac{\gamma}{\prod_{m=2}^{N-1} h_m}. \tag{21}$$

The solution to boundary value problem, puting (21) into (13) for the solution of problem (1),(18) we get

$$y_i = \frac{\gamma}{\prod_{m=2}^{N-1} h_m} \prod_{m=2}^{i-1} h_m = \frac{\gamma}{\prod_{m=i}^{N-1} h_m}, i \geq 3. \tag{22}$$

Thus the following theorem is proved.

Theorem 3. *Within the terms of Theorem 1 if the data α, β and $\gamma$of boundary conditions* (18) *are real constants satisfuing $\alpha \neq 0, \beta \neq 0$ and the conditions $g_i \neq 0, i \geq 1; h_i \neq 0, i \geq 2$ are fulfilled then boundary value problem* (1), (18) *has unique solution defined by expression* (21).

Now for the values $0 \leq i \leq N-3$ to the considered equation (1) *we assign the boundary condition below*

$$y_0 = \alpha, y_1 = \beta, y_N = \gamma, \tag{23}$$

where α, β and γ are the given real numbers (Pashavand & Aliyev, 2015).

Therefore, by putting the third boundary condition into the general solution for y_2 we obtain

$$\gamma = y_N = y_2 \prod_{m=2}^{N-1} h_m. \tag{24}$$

As result, they are determined by means of the dependent (11). Therefore, by writing the expressions (24) to (11), we get:

$$\gamma = y_2 \cdot \prod_{m=2}^{N-1} \left(\frac{y_2}{y_1} \prod_{s=1}^{m-1} g_s \right). \tag{25}$$

Here h_i are defined (11) dependendly on $y_1^{[1]}$.
So if we write (8) into of (25) and consider (23) we get

$$\gamma = y_2^{N-1} \cdot \prod_{m=2}^{N-1} \left(\beta^{-1} \prod_{s=1}^{m-1} \left(\frac{\alpha y_2}{\beta^2} \prod_{k=0}^{s-1} f_k \right) \right) =$$

$$= y_2^{N-1} \cdot \prod_{m=2}^{N-1} \left(\beta^{-1} y_2^{m-1} \prod_{s=1}^{m-1} \left(\alpha \beta^{-2} \prod_{k=0}^{s-1} f_k \right) \right) =$$

$$= y_2^{\frac{N(N-1)}{2}} \cdot \prod_{m=2}^{N-1} \left(\beta^{-1} \prod_{s=1}^{m-1} \left(\alpha \beta^{-2} \prod_{k=0}^{s-1} f_k \right) \right) =$$

$$= \frac{\alpha^{\frac{(N-1)(N-2)}{2}} y_2^{\frac{N(N-1)}{2}}}{\beta^{N(N-2)}} \cdot \prod_{m=2}^{N-1} \prod_{s=1}^{m-1} \prod_{k=0}^{s-1} f_k.$$

From the last we get

$$y_2^{\frac{N(N-1)}{2}} = \frac{\gamma \beta^{N(N-2)}}{\alpha^{\frac{(N-1)(N-2)}{2}} \cdot \prod_{p=2}^{N-1} \prod_{s=1}^{p-1} \prod_{k=0}^{s-1} f_k}.$$

It also implies

$$y_2 = \sqrt[\frac{N(N-2)}{2}]{\frac{\gamma\beta^{N(N-2)}}{\alpha^{\frac{(N-1)(N-2)}{2}} \cdot \prod_{p=2}^{N-1}\prod_{s=1}^{p-1}\prod_{k=0}^{s-1} f_k}} =$$

$$= \left(\frac{\gamma\beta^{N(N-2)}\alpha^{-\frac{(N-1)(N-2)}{2}}}{\prod_{p=2}^{N-1}\prod_{s=1}^{p-1}\prod_{k=0}^{s-1} f_k} \right)^{\frac{2}{N(N-1)}}. \tag{26}$$

Then the solution of boundary value problem (1), (23) can be obtained from (13) as follows

$$y_j = \left(\frac{\gamma\beta^{N(N-2)}\alpha^{-\frac{(N-1)(N-2)}{2}}}{\prod_{p=2}^{N-1}\prod_{s=1}^{p-1}\prod_{k=0}^{s-1} f_k} \right)^{\frac{2}{N(N-1)}} \cdot \prod_{m=2}^{j-1} h_m, j \geq 1. \tag{27}$$

Considering boundary conditions (23) and expressions (26) obtained for y_2 h_m are defined from (11) as follows

$$h_m = \frac{1}{\beta} \left(\frac{\gamma\beta^{N(N-2)}\alpha^{-\frac{(N-1)(N-2)}{2}}}{\prod_{p=2}^{N-1}\prod_{s=1}^{p-1}\prod_{k=0}^{s-1} f_k} \right)^{\frac{2}{N(N-1)}} \cdot \prod_{n=1}^{m-1} g_n . m \geq 2. \tag{28}$$

Here g_s are defined from (8) considering boundary conditions (23) and expression (26) obtained for y_2 as follows

$$g_s = \frac{\alpha}{\beta^2} \left(\frac{\gamma\beta^{N(N-2)}\alpha^{-\frac{(N-1)(N-2)}{2}}}{\prod_{p=2}^{N-1}\prod_{s=1}^{p-1}\prod_{k=0}^{s-1} f_k} \right)^{\frac{2}{N(N-1)}} \cdot \prod_{k=0}^{s-1} f_k, s \geq 1. \tag{29}$$

Thus the following result is proved.

Theorem 4. *Within the terms of Theorem 1, if the data α, β and γ of the boundary condition are real constants, the condition $f_k \neq 0$ is fulfilled and* (26)-(29) *are defined, then there exists a solution to problem* (1), (23) *given in the form* (27).

Note 1. The solution we obtained for the boundary value problem is not unique. So that, the solution (26) we obtained for y_2 is not uniqie and is obtained from the algebraic equation of order $\frac{N(N-1)}{2}$. By virtue of the main theorem of algerba this algebraic equation has number of slutions.

3 Conclusion

In the paper the Cauchy and boundary value problems for the third order equation with discrete multiplicative derivatives are considered. The analytical expressions are obtained for the solutions of these problems.

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